Poisson boundaries for random walks on groups

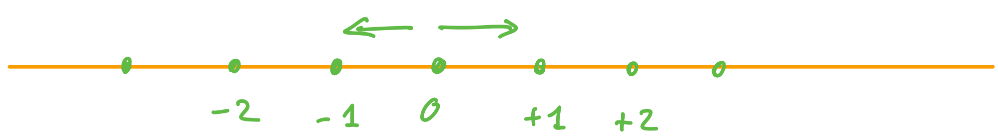
– with a view towards geometric group theory

Lecture 1
1) introduction to random walks
2) definition of Poisson boundary

Lecture 2
3) entropy theory
4) identification criteria

Lecture 3
5) applications to geometric group theory
6) appl. to Sublinearly Morse boundary

E.g.: 1 $\quad G = \mathbb{Z} \quad , \quad \mu = \dfrac{\delta_{+1} + \delta_{-1}}{2}$

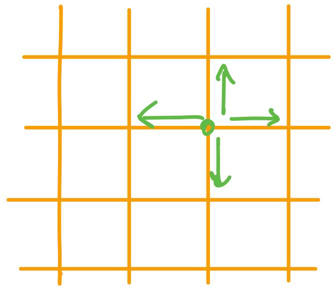


$w_n = X_1 + \dots + X_n$ where (X_i) are i.i.d.

$\uparrow$ increment

RW is recurrent: $\mathbb{P}(w_n = 0 \text{ infinitely often}) = 1$

② $G = \mathbb{Z}^d$ d-dimensional grid



For $d = 2$: RW is recurrent

For $d \geq 3$: RW is transient

$\mathbb{P}(w_n = 0 \text{ infinitely often}) = 0$

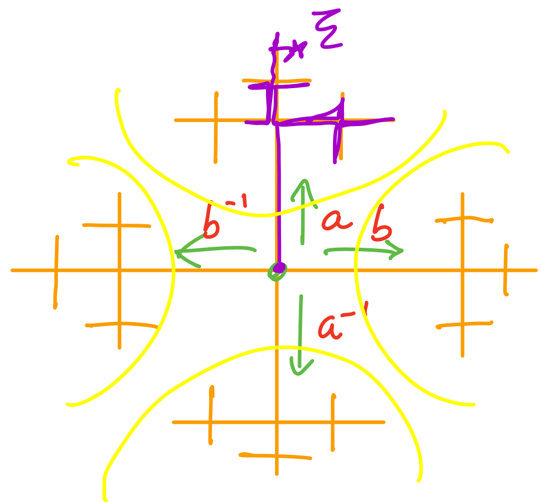
Def: The DRIFT/SPEED of RW is

$$\ell = \lim_{n \to \infty} \frac{d(w_n, o)}{n} \quad \text{almost surely}$$

For symmetric RW on $\mathbb{Z}^d$: $\ell = 0$

③ $G = \mathbb{F}_2 = \langle a, b \rangle$ $\mu = \frac{1}{4}\left(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}}\right)$

① The drift $\ell > 0$



Idea

$$d_n := d(w_n, o)$$

$\mathbb{E}[d_n] \geq \frac{n}{2}$ [exercise]

② A.e. sample path converges to the Gromov boundary.

Definition of RW

Let G be countable group, let μ be prob. measure on G.

Consider sequence (g_n) of elements of G, independent, identically distributed with distribution μ. .

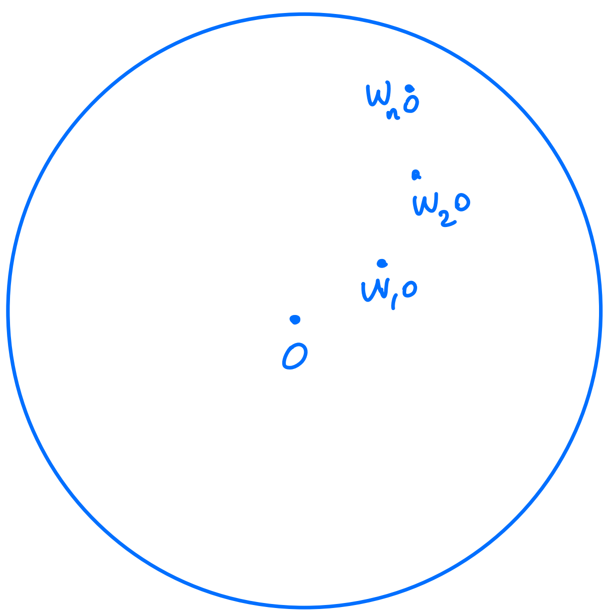
$$w_n = g_1 g_2 \cdots g_n$$

Ex.: $\mu = \frac{1}{2}(\delta_a + \delta_b)$ $\quad w_n = a\,a\,b\,b\,a\,b$

(g_n) = INCREMENTS i.i.d.

(w_n) = SAMPLE PATH NOT i.i.d.

Let $G < \mathrm{Isom}(X, d)$, pick $o \in X$ base point



(4) $G < PSL_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}) = \mathrm{Isom}^+(\mathbb{D})$

$\mu = \frac{1}{2}(\delta_A + \delta_B)$

$w_n = ABBAB\ldots$

Two spaces of sequences

Step space (increments) $\quad (G^{\mathbb{N}}, \mu^{\mathbb{N}}) \ni (g_n)$ IID

$$(G^{\mathbb{N}}, \mu^{\mathbb{N}}) \xrightarrow{p} (G^{\mathbb{N}} = \Omega, \mathbb{P})$$

$$(g_n) \longmapsto (w_n) = (g_1 \cdots g_n)$$

$$\mathbb{P} := p_*(\mu^{\mathbb{N}})$$

Two shifts

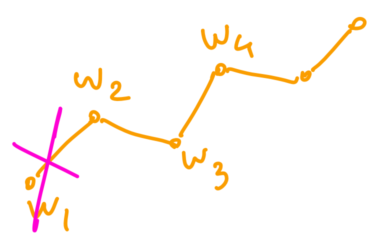
$\sigma : G^{\mathbb{N}} \to G^{\mathbb{N}}$ measure preserving

$\sigma(g_n) = (g_{n+1})$

$T : \Omega \to \Omega$

$T(w_n) = (w_{n+1})$

"time shift, leaving the location of the walk fixed"

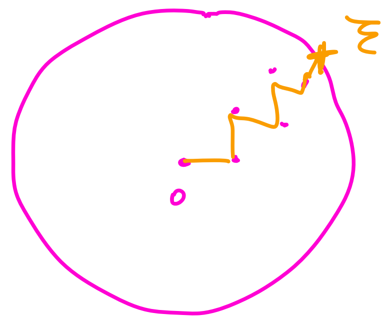


$$w_n = g_1 g_2 \cdots g_n \overset{\sigma}{\rightsquigarrow} w_n' = g_2 \cdots g_n = g_1^{-1} w_n$$

$\omega = (g_n)$

Questions

(1) Does random walk converge (almost surely) to a suitable ∂X ?



Example

Lemma (Furstenberg)

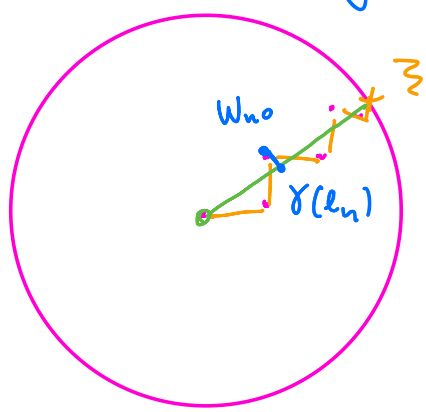
If μ is a non-elementary measure on $PSL_2(\mathbb{R})$, then for a.e. (w_n) the limit

$$\xi = \lim_{n\to\infty} w_n o \in \partial \mathbb{D} \quad \underline{\text{exists}}$$

<u>non-elementary</u>: $\langle \operatorname{supp}\mu \rangle$ not contained in 1-parameter subgroup

(2) How "good" is the convergence? E.g.: are sample paths close to geodesic rays in the space X ?

(sublinear tracking – ray approximation)



SUBLINEAR TRACKING

$\exists\, \gamma : [0,\infty) \to X$ s.t.

$$\frac{d(w_n o, \gamma(\ell_n))}{n} \to 0$$

③ What are the properties of the hitting measure? E.g.: ⊛ is it the same as the Lebesgue / Patterson-Sullivan measure?

Def.: If a.s. $(w_n o)$ converges to ∂X, define HITTING MEASURE ν_μ

$$\nu_\mu(A) = \mathbb{P}\left(\lim_n w_n o \in A\right)$$

④ Is there a Poisson representation formula?

Duality between:

$$\left\{\begin{matrix}\text{bounded harmonic}\\ \text{functions on } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{bounded measurable}\\ \text{functions on } \partial G\end{matrix}\right\}$$

$\boxed{Q}$ Given a RW on $G < \mathrm{Isom}(X)$ with a notion of ∂X s.t. the RW converges a.s. to ∂X, do we have a Poisson Rep formula on ∂X? (Poisson boundary)

The Poisson (- Furstenberg) boundary

Poisson representation formula

HARMONIC FUNCTIONS

$h^\infty(\mathbb{D}) := \{u : \mathbb{D} \to \mathbb{R},\ \Delta u = 0,\ \sup|u| < \infty\}$

Thm There is a correspondence

$$h^\infty(\mathbb{D}) \longleftrightarrow L^\infty(\partial\mathbb{D})$$
$$\varphi_f$$

1) $\longrightarrow$ take limit value

$$f(\xi) = \lim_{z \to \xi} u(z)$$

probabilistic interpretation

$$f(\xi) = \lim_{t \to \infty} \mathbb{E}\left[u(B_t) \,\middle|\, B_\infty = \xi \right]$$

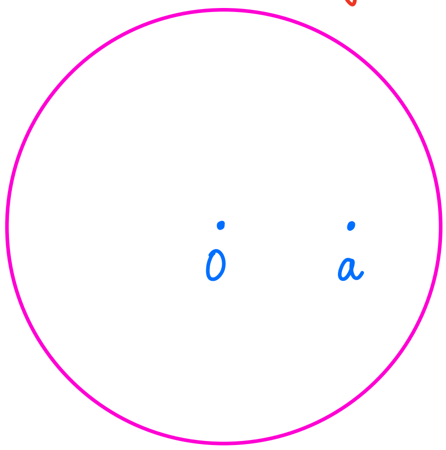
2) ← convolution with Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) P_r(t - \theta)\, dt$$

Then u is harmonic.

Interpretation of the Poisson rep. formula



$$\mathbb{D} \simeq SL_2(\mathbb{R}) / SO_2(\mathbb{R})$$

$$a = re^{i\theta}$$

There exists $g \in \mathrm{Isom}(\mathbb{D})$ s.t. $g(0) = a$

$$g(z) = \frac{a - z}{1 - \bar{a} z} \qquad |g'(z)| = \frac{1 - |a|^2}{|1 - \bar{a} z|^2}$$

$$z = e^{it}$$

$$|g'(e^{it})| = \frac{1 - r^2}{|1 - re^{i(t-\theta)}|^2}$$

$$u(re^{i\theta}) = \int_{-\pi}^{\pi} f(e^{it}) \, |g'(e^{it})| \, \frac{dt}{2\pi}$$

$$= \int_{\partial \mathbb{D}} f(\xi) \, \frac{dg\lambda}{d\lambda}(\xi) \, d\lambda(\xi)$$

$$= \int_{\partial \mathbb{D}} f(\xi) \, dg\lambda(\xi)$$

Harmonic functions on groups

Let (G,μ) be measured group.

Def.: A function $f: G \to \mathbb{R}$ is μ-HARMONIC if

$$f(g) = \int_G f(gh)\, d\mu(h)$$

$$H^\infty(G,\mu) := \{ f: G \to \mathbb{R},\ \text{bounded},\ \mu\text{-harmonic} \}$$

Def.: A measure ν on M is μ-STATIONARY if

$$\nu = \sum_g \mu(g)\, g_* \nu$$

"invariant on average"

Lemma

If the RW converges to ∂X almost surely, then the hitting measure is μ-stationary.

Proof

$$\nu(A) = \mathbb{P}\left(\lim_{n\to\infty} w_n o \in A \right)$$

$$= \sum_g \mathbb{P}(g_1 = g)\, \mathbb{P}\Big(\lim_{n\to\infty} \overset{g_1 g_2 \cdots g_n o}{w_n o} \in A \;\Big|\; g_1 = g\Big)$$

$$= \sum_g \mu(g)\, \underline{\mathbb{P}\big(\lim_n g_2 \cdots g_n o \in g^{-1}A\big)}$$

$$= \sum_g \mu(g)\, \nu(g^{-1}A)$$

$$d(w_n x \;,\; w_n y) = d(x, y)$$

Def.: A measure space (B, ν) on which G acts by homeos is a μ-boundary if there is a G-equiv. map

$$\mathrm{bnd} : \Omega \longrightarrow B \qquad \text{s.t.}$$

$$\mathrm{bnd} = \mathrm{bnd} \circ T .$$

E.g.: if RW converges to ∂X, define

$$\mathrm{bnd}(\omega) := \lim_n w_n o \in \partial X$$

$$\mathrm{bnd} \circ T(\omega) = \lim_n w_{n+1} o = \mathrm{bnd}(\omega)$$

Rmk: For every (G, μ), $(\{pt\}, \delta_0)$ is a μ-bdry.

We want: $$H^\infty(G, \mu) \overset{?}{\longleftrightarrow} L^\infty(B, \nu)$$

Def.: The Poisson transform is

$$P : L^\infty(B,\nu) \longrightarrow H^\infty(G,\mu)$$
$$f \mapsto Pf(g) := \int_B f \, g_*\nu$$

Def.: A μ-boundary (B,ν) is a model for the Poisson boundary if the Poisson transform
$$P : L^\infty(B,\nu) \longrightarrow H^\infty(G,\mu)$$
is an isomorphism.

R1 $(B,\nu) \simeq (\{1\}, \delta_0)$ trivial

$\Updownarrow$

Every bdd harmonic function is constant

LIOUVILLE PROPERTY

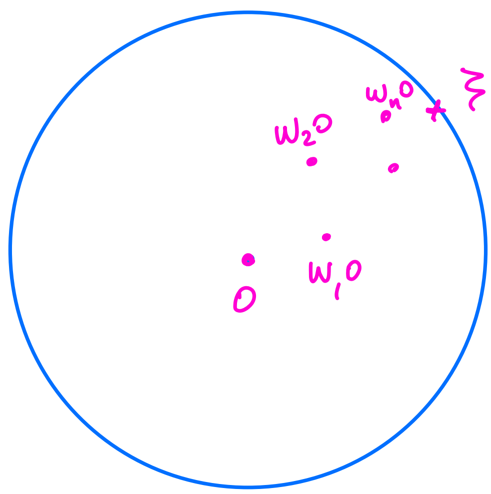
Poisson boundaries for RWs on groups

Lecture 2 Entropy Theory & Identification

Kaimanovich-Vershik; Derriennic

Let (G,μ) measured group, $G < \mathrm{Isom}(X,d)$

$w_n = g_1 \cdots g_n$



Suppose:

A.e. sample path converges to ∂X

$$\xi = \lim_n w_n o$$

$$\nu_\mu(A) = \mathbb{P}\left(\lim_n w_n o \in A\right)$$ HITTING MEASURE

Then:

$(\partial X, \nu_\mu)$ is a μ-boundary

$\boxed{Q}$ Is $(\partial X, \nu_\mu)$ a model for the Poisson boundary ?

bounded harmonic ? measurable on ∂X

$$H^\infty(G,\mu) \xrightarrow{\sim} L^\infty(\partial X, \nu_\mu)$$

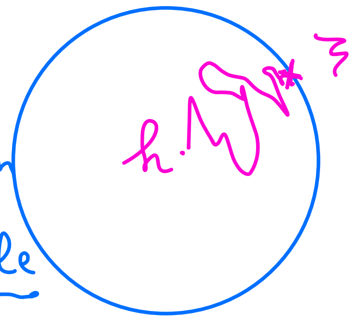
Poisson transform ($\leftarrow$)

$f \in L^\infty(\partial X)$ $\qquad (Pf)(g) = \int f \, g_* \nu$

Inverse Poisson transform

($\rightarrow$)

$h \in H^\infty(G,\mu)$ $\qquad \mathbb{E}[X_{n+1} | X_n] = X_n$



$X_n := h(w_n)$ is a martingale

$$\mathbb{E}\left[h(w_{n+1}) \,\middle|\, w_n = g\right] = \sum h(\underbrace{w_n g_{n+1}}_{w_{n+1}})\mu(g_{n+1}) = h(w_n)$$

$\Lambda(h)(\xi) := \lim_{n\to\infty} h(w_n)$ if w_n

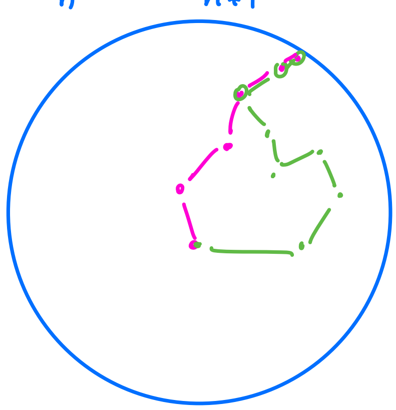
Construction of Poisson boundary

Def.: The Poisson equivalence class is defined on Ω as:

$\omega \sim \omega'$ if $\exists\, m,n$ s.t. $T^m(\omega) = T^n(\omega')$

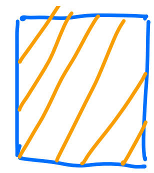
$T(w_n) = w_{n+1}$ $\qquad$ Poisson equiv. $w_{m+k} = w'_{n+k}$ $\forall k$

Fig.:



tail equiv. $w_{n+k} = w'_{n+k}$ $\forall k$

irrational foliation



Consider $p: \Omega \longrightarrow \Omega/\sim$

Problem $\Omega/\sim$ is not a "nice" Borel space

A bit of measure theory

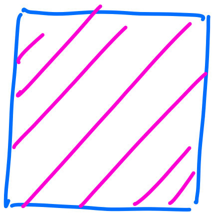
Def.: A Borel space $(X, \mathcal{A})$ is standard if it is iso to (M, Borel) with M compact metrizable

Thm: $(X, \mathcal{A}) \simeq [0,1] \cup \bigcup_n \{p_n\}$

Def.: A Borel space is countably separated if $\exists$ collection $(B_n)_{n \in \mathbb{N}}$ of measurable subsets s.t. $\forall x \neq y$, $\exists B_n$ s.t. $x \in B_n$, $y \notin B_n$

E.g.: standard $\longrightarrow$ countably separated

Non-example Take $T(x) = x + \alpha \pmod 1$ $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $X/_T$ is not countably separated

Fig.: 

Suppose $(B_n)_{n\in\mathbb{N}}$ separating

$$P_n = \bigvee_{k=1}^{n} (B_k, B_k^c)$$

$\forall n \ \exists C_n \in P_n : \mu(C_n) = 1$ $\qquad C_{n+1} \subseteq C_n$

$C = \bigcap C_n$, $\underline{\mu(C) = 1}$. B_n sep. $\Longrightarrow$ $C = \{1 \text{ orbit}\}$

contradiction

Def.: A measure space $(X, \mathcal{A}, \mu)$ is Lebesgue if it contains a full measure subset which is standard Borel.

Def.: A partition ξ on $(X, \mathcal{A}, \mu)$ is measurable if X/ξ is countably separated.

Thm (Rokhlin)

Given a partition ξ of $(X, \mathcal{A}, \mu)$ in measurable sets, there exists the finest partition $\hat{\xi}$ which refines to ξ and which is measurable. $\hat{\xi}$ is called the MEASURABLE ENVELOPE. Then the quotient

$$(X/\hat{\xi}, \mathcal{A}/\hat{\xi}, p_*\mu) \text{ is}$$

a Lebesgue space.

Def.: The SPACE OF ERGODIC COMPONENTS is $X /\!\!/ T$ the quotient of X by the measurable envelope of $\overset{T}{\sim}$.

Construction of Poisson boundary

Def.: The POISSON BOUNDARY of (G,μ) is
$$(B_{PF}, \nu_{PF}) := (\Omega, \mathbb{P}) /\!\!/ \overset{T}{\sim} .$$

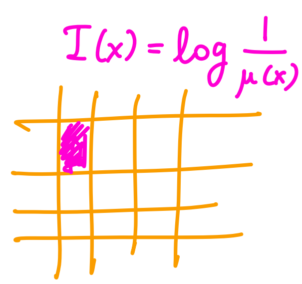
We have boundary map $\mathrm{bnd} : (\Omega, \mathbb{P}) \to (B_{PF}, \nu_{PF})$

Cor.:
$$L^\infty(B_{PF}, \nu_{PF}) \cong L^\infty(\Omega, \mathbb{P})^T$$
← T-invariant

Universal Property

For any G-equivariant $f : (\Omega, \mathbb{P}) \longrightarrow (Y, \lambda)$
s.t. $f \circ T = f$, there exist
$g : (B_{PF}, \nu_{PF}) \longrightarrow (Y, \lambda)$ s.t.

$$\begin{tikzcd}
(\Omega, \mathbb{P}) \arrow[loop above, "T"] \arrow[r, "f"] \arrow[d, "\mathrm{bnd}"] & (Y, \lambda) \\
(B_{PF}, \nu_{PF}) \arrow[ur, dashed, "g"] &
\end{tikzcd}$$



Entropy Theory

Let μ be measure on G.

Def.: $H(\mu) := -\sum_g \mu(g) \log \mu(g)$

Let $\mu_n := \underbrace{\mu * \dots * \mu}_{n \text{ times}}$

$H(\mu_{n+m}) \leq H(\mu_n) + H(\mu_m)$

Fekete

$\mu_n(g) = \mathbb{P}(w_n = g)$

Def.: The ASYMPTOTIC (AVEZ) ENTROPY is

$$H_\infty(\mu) := \lim_{n\to\infty} \frac{H(\mu_n)}{n}$$

"amount of information gained from one step to the next"

THE ENTROPY CRITERION

Thm (Kaimanovich-Vershik; Derriennic; Rosenblatt)
Suppose $H(\mu) < \infty$. Then

Poisson boundary is trivial $\overset{\text{iff}}{\longleftrightarrow}$ $H_\infty(\mu) = 0$

Cor.: if G has subexponential growth then its Poisson boundary is trivial for any finitely supported μ.

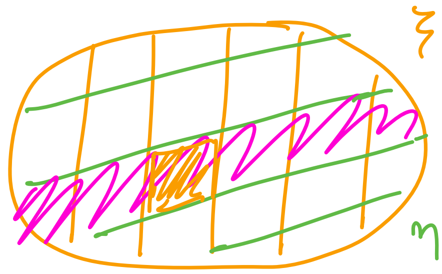
$$\frac{1}{n} H(\mu_n) \le \frac{1}{n} \log \#\underbrace{\{g : \|g\| \le Cn\}}_{\mathrm{supp}\,\mu_n} \to 0$$

Proof Let ξ, η partitions. Define the RELATIVE ENTROPY

$$H(\xi | \eta) = -\int \log \frac{m(\xi(x) \cap \eta(x))}{m(\eta(x))} \, dm(x)$$

Note: $H(\xi|\eta) = 0$ iff ξ, η are independent.

Fig.



Partitions

α_k : $g_n = g_n'$ for $n \le k$ head partition

η_k : $w_n = w_n'$ for $n \ge k$ tail partition

$$\boxed{\eta_\infty = \bigwedge_n \eta_n}$$ Poisson partition

← coarser than η_n $\forall$

Claim

$$H(\alpha_k \mid \eta_n) = k\, H_1 + H_{n-k} - H_n \qquad (0 \le k \le n)$$

Proof $H_n = H(\mu_n)$

$$\mathbb{P}(\alpha_k = (g_1, \dots, g_k) \mid \eta_n = g) = \frac{\mu(g_1)\cdots\mu(g_k)\, \mu_{n-k}(g_k^{-1}\cdots g_1^{-1} g)}{\mu_n(g)}$$

Since $\eta_{n+1} \leq \eta_n$,

$$H(\alpha_1 | \eta_\infty) = \lim_{n\to\infty} H(\alpha_1 | \eta_{n+1})$$

Hence

$$H(\alpha_1 | \eta_\infty) = H_1 + \lim_{n\to\infty} \left(H_{n-1} - H_n\right)$$

asymptotic h

$$= H(\mu) - H_\infty(\mu)$$

$$H(\alpha_k | \eta_\infty) = k H(\mu) - k H_\infty(\mu)$$

So : $H_\infty(\mu) = 0$

$\Updownarrow$

$H(\alpha_k | \eta_\infty) = H(\alpha_k)$ $\forall k$

$\Updownarrow$

α_k, η_∞ independent $\forall k$

$\Updownarrow$

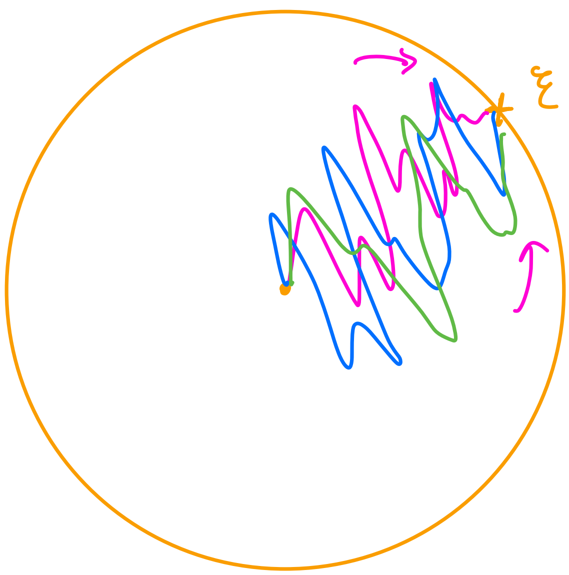
η_∞ = trivial (modulo null sets)

Relative Entropy

Suppose RW converges to ∂X a.s.

Let ν be hitting measure. Let $\xi \in \partial X$.

Fig. 1



$$\mathbb{P}(w_n = g \mid w_\infty \in A) = \qquad w_\infty = w_n u_\infty$$

$$= \mathbb{P}(w_n = g \mid u_\infty \in g^{-1}A)$$

$$= \frac{\mathbb{P}(w_n = g)\, \nu(g^{-1}A)}{\nu(A)} = \mu_n(g) \frac{g\nu(A)}{\nu(A)}$$

Def.: The CONDITIONAL RANDOM WALK at ξ is the Markov process on G defined as

$$\mu_n^{(\xi)}(g) = \underbrace{\mu_n(g)}_{\text{original RW}} \underbrace{\frac{dg\nu}{d\nu}(\xi)}_{\text{RN derivative of hitting meas.}}$$

Def.: The RELATIVE ENTROPY at ξ is

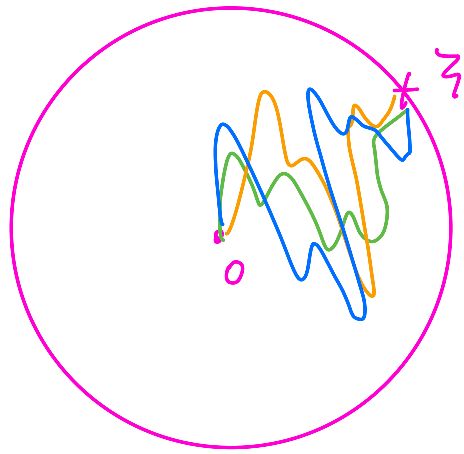
$$H_\infty^{(\xi)} := \lim_{n\to\infty} \frac{H(\mu_n^{(\xi)})}{n}$$

$\leftarrow$ distribution of RW conditioned to hitting ξ at

Thm (Relative Entropy Criterion)
Let (B,ν) be a μ-boundary for (G,μ), with $H(\mu)<\infty$. Then (B,ν) is the Poisson boundary iff

$$H_\infty^{(\xi)} = 0 \quad \text{for } \nu\text{-a.e. } \xi \in B.$$

Fig



"relative entropy" is zero

↓

boundary cannot be "further split"

Def.: The measure μ has FINITE FIRST MOMENT if $\int d(go,o)\, d\mu(g) < \infty$.

Def.: $G < \mathrm{Isom}(X)$ has EXPONENTIALLY BOUNDED GROWTH if $\exists\, C$:

$$\#\{g \in G : d(o,go) \leq R\} \leq C e^{CR}$$

Thm (RAY APPROXIMATION CRITERION – Kaimanovich)

Let:
- G be a countable group
- $G < \mathrm{Isom}(X, d)$ an action of exponentially bounded growth
- μ a measure with finite 1st moment
- (B, ν) a μ-boundary

Suppose that there are maps

$\pi_n : B \longrightarrow G$ such that

$$\lim_{n \to \infty} \frac{d(w_n, \pi_n(\mathrm{bnd}(\omega)))}{n} = 0 \quad \text{a.s.}$$

Then (B, ν) is the Poisson boundary

Poisson boundaries for RWs on groups

Lecture 3: Applications to geometric group theory

$G < \mathrm{Isom}(X, d)$

μ on G, $\quad H(\mu) < \infty \quad , \quad \int_G d(o, go)\, d\mu(g) < \infty$

Examples

G abelian $\longrightarrow \partial_p(G,\mu)$ trivial
(Blackwell, Choquet-Deny)

G nilpotent $\longrightarrow \partial_p(G,\mu)$ trivial
(Dynkin-Malyutov)

G subexp growth $\longrightarrow \partial_p G$ trivial
(Kaimanovich-Vershik)

G nonamenable $\longrightarrow \partial_p(G,\mu)$ not trivial $\forall \mu$

G amenable $\longrightarrow \exists \mu : \partial_p(G,\mu)$ trivial
(Kaimanovich-Vershik)

(Erschler; Frisch-Hartman-Tamuz-V.Ferdowsi)

$H_\infty(\mu) = 0 \longleftrightarrow \ell(\mu) = 0$ μ symmetric

DRIFT (Karlsson - Ledrappier)

Cor.: G nonamenable $\to H_\infty(\mu) > 0 \to \ell(\mu) > 0$

1) G hyperbolic group, $X = \mathrm{Cay}(G, S)$

∂X = Gromov boundary

$(B_{PF}, \nu_{PF}) \simeq (\partial X, \nu)$ (Kaimanovich)

2) G rel hyperbolic group (Gautero-Mathéus) (Qing-Rafi-T)

3) $G < \mathrm{Isom}(X, d)$ X CAT(0)

∂X = visual (Karlsson-Margulis)

4) $G = \mathrm{Mod}(S)$ mapping class group

$X = \mathrm{Teich}(S)$ $\partial X = PML$ (Kaimanovich Masur)

$X = \mathcal{C}(S)$ curve complex $\partial X = \underbrace{\partial_{hyp}}_{\text{Gromov bdry}} \mathcal{C}(S)$ (Maher T)

$$\boxed{X = \mathrm{Cay}(\mathrm{Mod}(S)) \quad \partial X = \text{sublinearly Morse boundary} \quad \text{(Qing-Rafi-T)}}$$

5) $G = Out(F_n)$

X = Outer Space CV_n $\quad \partial X = \partial CV_n$ (Horbez)

X = Free factor complex F_n $\quad \partial X = \partial_{hyp} F_n$

6) $G < Isom(X, d)$, $\quad X$ δ-hyperbolic

If action has one WPD element, then

$(\partial X, \nu)$ is P.B. $\quad$ (Maher-T)

Thm (Gekhtman-Qing-Rafi-T.

Let G be f.g. group, (X, d) Cayley graph

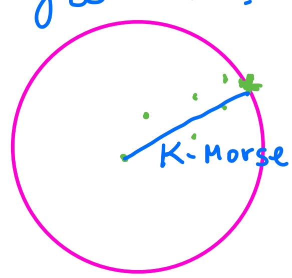
Let μ on G with finite 1^{st} moment,

Suppose $\langle supp\, \mu \rangle$ is non-amenable.

Let $K(r)$ concave, sublinear.

Suppose: for a.e. sample path ω there exists a K-Morse geodesic ray γ_ω s.t.

$$\lim_{n\to\infty} \frac{d(w_n, \gamma_\omega)}{n} = 0$$



Then:

① a.e. sample path converges to $\partial_K X$

② $(\partial_\kappa X, \nu)$. is Poisson boundary.

Thm (Qing-Rafi-T.).
Let G be relatively hyperbolic, and $\boxed{\kappa(t) = \log(t)}$. Then for any finitely supported μ on G,

$$(\partial_\kappa G, \nu) \approx \partial_p(G, \mu).$$

Thm (Qing-Rafi-T.)
Let G be mapping class group of $S_{g,b}$.

Let $\kappa(t) = \log^p(t)$, $p = 3g-3+b$. ← depth of hierarchy

Then for any finitely supported μ on G,

$$(\partial_\kappa G, \nu) \approx \partial_p(G, \mu).$$

SUBLINEAR TRACKING PROPERTY

Cor.: If $\overline{X} = X \cup \partial X$ is bordification of X,

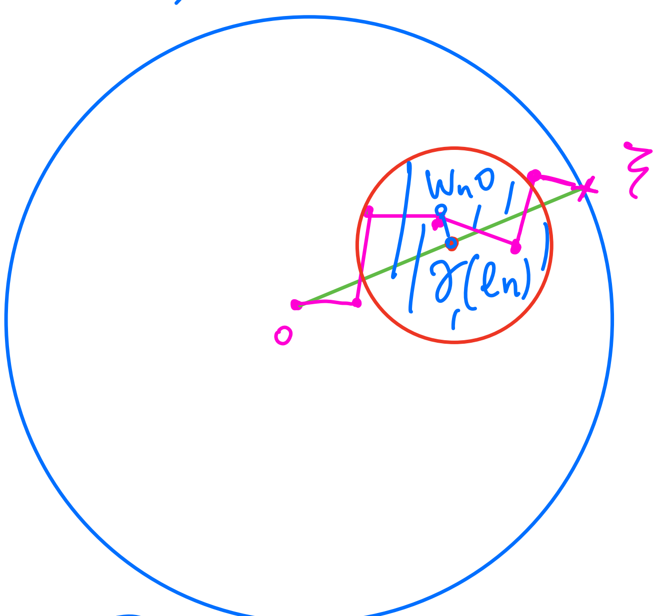
- a.e. $(w_n o)$ converges to ∂X
- for a.e. $\xi \in \partial X$ there exists a

quasi-geodesic ray $\gamma : [0, \infty) \longrightarrow X$ s.t.

$$\lim_{n \to \infty} \frac{d(w_n o, \gamma(\ell_n))}{n} = 0 \quad \text{a.s.},$$

Then $(\partial X, \nu)$ is the Poisson boundary.

Proof



Sublinear Tracking

$\Downarrow$

Relative Entropy is 0 a.s.

∂X is P.B. $\longleftrightarrow$ For a.e. $\xi \in \partial X$
$H_\infty(\mathbb{P}^\xi) = 0$

$\#\{g : d(o, go) \le R\} \le 6 e^C$

$$H(\mu_n^{(\xi)}) \leq \log \# \operatorname{supp} \mu_n^{(\xi)}$$
$$\leq C\, d(w_n o, \gamma(\ell n))$$

so

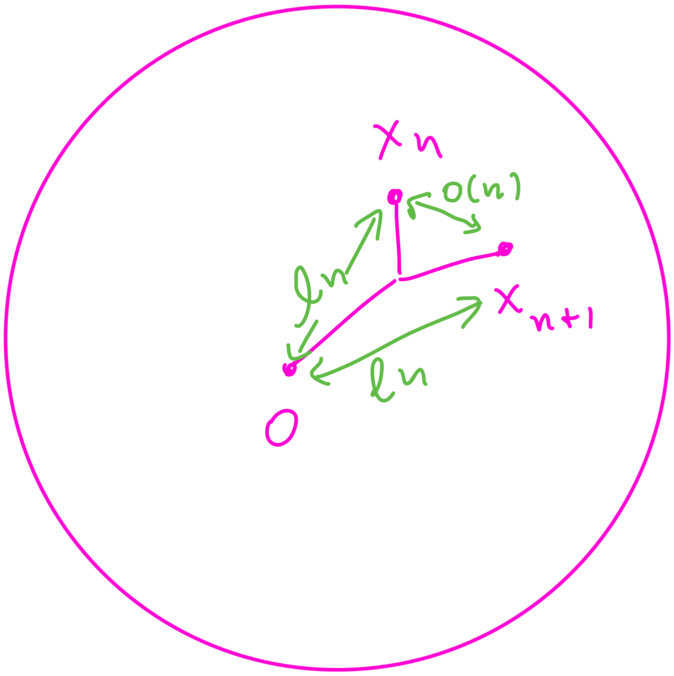
$$\frac{H(\mu_n^{(\xi)})}{n} \leq C \, \frac{d(w_n o, \gamma(\ell n))}{n} \longrightarrow 0$$

Hyperbolic groups

Lemma (Delzant)

(x_n) sublinearly tracks a geodesic iff ① $d(x_n, x_{n+1}) = o(n)$

② $\frac{|x_n|}{n} \to \ell$



$$(x_n, x_{n+1})_o = \frac{1}{2}\left(|x_n| + |x_{n+1}| - d(x_n, x_{n+1})\right)$$

$$\geq n\ell + o(n)$$

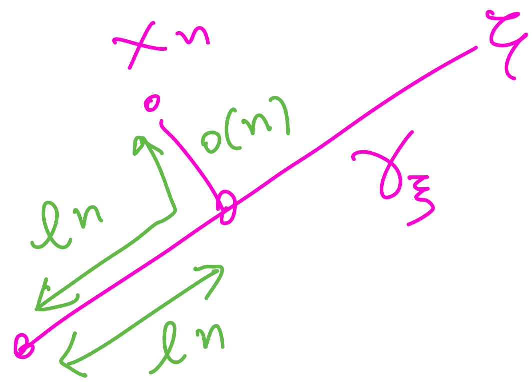
$$d_{vis}(x_n, x_{n+1}) \lesssim e^{-n\ell}$$

$$\Downarrow$$

$\lim_n x_n = \xi \in \partial X$ exists

$$d_{vis}(x_n, \xi) \lesssim e^{-n\ell}$$

Hence $d(x_n, \gamma_\xi(\ell n)) = o(n)$



<u>Thm</u> If (G,μ) is finite 1^{st} moment RW on non-elementary hyperbolic group G, then for a.e. (w_n) there exists γ s.t.

$$\frac{d(w_n, \gamma(\ell n))}{n} \to 0$$

Hence, $(\partial G, \nu)$ is Poisson boundary.

Proof Since G non-amenable, $H_\infty(\mu)>0$ hence $\ell(\mu)>0$. Since μ has finite first moment, $d(w_n, w_{n+1}) = d(1, g_{n+1})$ satisfies

$$\frac{d(1, g_{n+1})}{n} \to 0 \quad a.s.$$

Hence apply lemma,

$$f(\omega) := d(o, g_1 o) \qquad f : \Omega \to \mathbb{R}, \quad \sigma \curvearrowright \Omega$$

Then

$$\lim_{n\to\infty} \frac{f(\sigma^n \omega)}{n} = 0 \quad a.s.$$

Geometric Lemmas

Lemma (convergence criterion)
Let γ be a κ-Morse quasigeodesic based at $o \in X$. Let $(x_n) \subseteq X$ a sequence, with $\|x_n\| \to \infty$. Suppose there exists C s.t.

$$d(x_n, \gamma) \leq C \cdot \underline{\kappa}(\|x_n\|).$$

Then (x_n) converges to $[\gamma]$ in $X \cup \partial_\kappa X$.

Application to the mapping class group

S = surface, $\mathcal{C}(S)$ = curve complex of S

d_S = distance in $\mathcal{C}(S)$

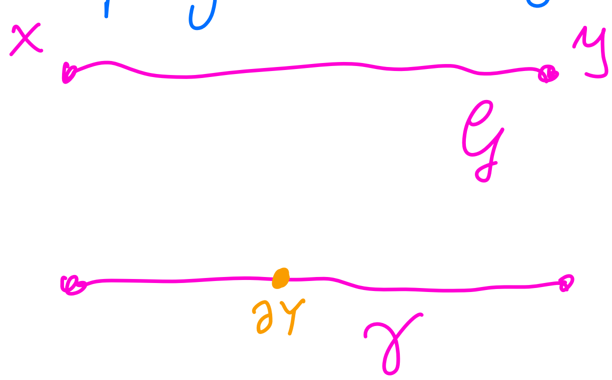
$\partial\mathcal{C}(S) = \mathcal{EL}$ ending laminations

$Y \subseteq S$ subsurface $\longrightarrow (\underbrace{\mathcal{C}(Y)}_{\text{curve complex of } Y}, d_Y)$

Clouds

Given $x, y \in \mathrm{Mod}(S)$:

1) hierarchy path $\mathcal{G}$ in $\mathrm{Mod}(S)$
2) projection γ to $\mathcal{C}(S)$



$$\begin{tikzcd} \mathrm{Mod}(S) \arrow[d] \arrow[dr] & \\ \mathcal{C}(S) & \mathcal{C}(Y) \end{tikzcd}$$

logarithmic bounded projection

$$\mathcal{L}_c = \{\xi \in \mathcal{EL} : d_Y(o, \xi) \leq c \log d_S(o, \partial Y) \ \forall Y\}$$

cloud = "lifts" of γ to $\mathrm{Mod}(S)$

$$\mathcal{Z}(o, \xi) = \{ z \in \mathrm{Mod}(S) : d_Y(z_Y, [o, \xi]) \leq D \ \forall Y\}$$

Eskin-Masur-Rafi

barycenter in $\mathrm{Mod}(S)$ Behrstock-Minsky

$$\mathcal{C}(S) \longrightarrow \mathcal{C}(Y)$$

Lemma If $\xi \in \mathcal{L}_c$, then any resolution of a hierarchy G_ξ is K-contracting, hence K-Morse.

$K(r) = \log^p(r)$

Step 1 For any k there is C s.t.

$$\mathbb{P}\left(\sup_Y d_Y(1, w_n) \geq C \log n\right) \leq C n^{-k}$$

Pf Linear Progress w. Exponential Decay (Maher)

$$\mathbb{P}\left(d_S(1, w_n) \leq \tfrac{\ell}{2} n\right) \leq C e^{-n/C}$$

Bounded geodesic image theorem

if $d_S([x,y], \partial Y) \geq C$, then $d_Y(x,y) \leq C$

Fig

Set $n \to A \log n$

$$\mathbb{P}\left(d_S(1, w_{A\log n}) \leq 2\right) \leq C n^{-A/C}$$

$$d_G(x,y) \lesssim C_2 \sum_K \left[d_K(x,y)\right]_B + C_2$$

$$\sup_Y d_Y(1, w_n) \geq \log n$$

$$\Downarrow$$

$\exists\, i_1, i_2 : |i_2 - i_1| \geq \log n$ s.t.

$$d_S([w_{i_2}, w_{i_1}], \partial Y) \leq 2$$

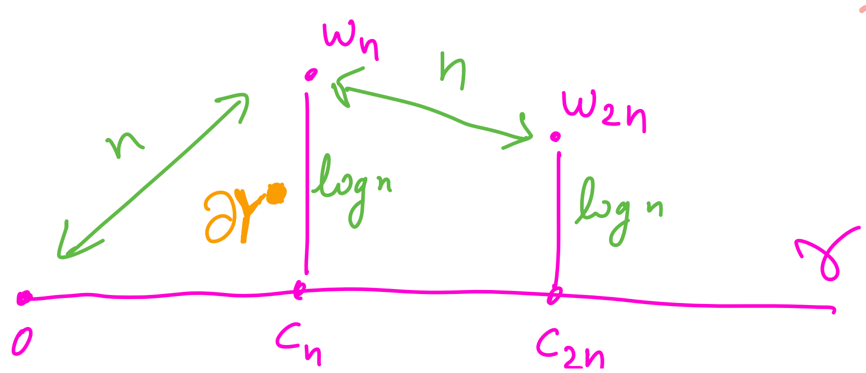
choose i_1, i

hence $\mathbb{P}\left(\sup_Y d_Y(1, w_n) \geq A \log n\right) \leq n^2 n^{-A/C}$

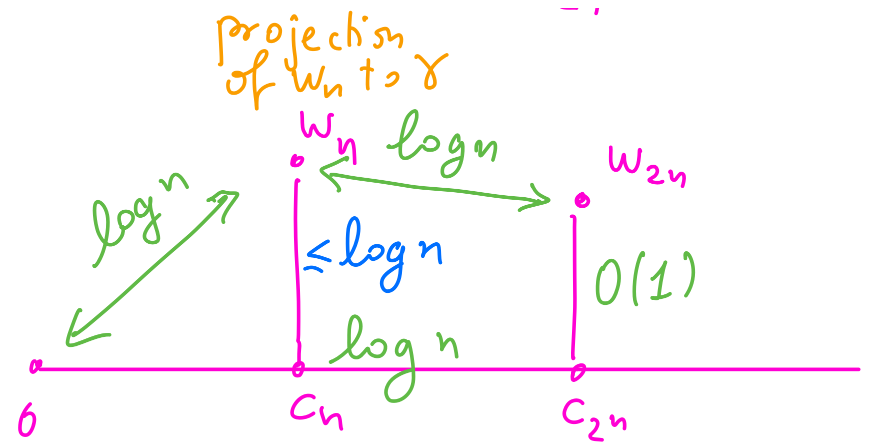
$$\mathbb{P}\left(\sup_Y d_Y(w_{i_1}, w_{i_2}) \geq A \log n\right) \leq n^{-A/C}$$

Step 2 Given k, there is C s.t. for a.e. $\omega \in \Omega$,

$$\mathbb{P}\left(\sup_Y d_Y(1, w_n) \geq C \log \underbrace{d_S(1, w_n)}_{\approx n}\right) \leq n^{-k}$$



$\mathcal{G}(S)$



$\mathcal{C}(Y)$

<u>Claim</u> $\mathbb{P}\left(\sup_Y d_Y(w_n, c_n) \gtrsim C \log n\right) \leq n^{-2}$

if $d_Y(w_n, c_n) \gg 1$, then

$$d_S([w_n, c_n], \partial Y) \leq 1.$$

But then: $d_S([w_{2n}, c_{2n}], \partial Y) \gg 1$

so $d_Y(w_{2n}, c_{2n}) \lesssim 1$

so $d_Y(w_n, c_n) \lesssim \log n$ triangle inequality

<u>Step 3</u> $\mathbb{P}\left(\sup_Y d_Y(1, w_n) \gtrsim C \log n\right) \leq n^{-2}$

Borel - Cantelli:

For a.e. ω,

$$\sup_Y d_Y(1,w_n) \le 6 \log n \qquad \text{for } n \ge n_0$$

Positive Drift: $\mathbb{P}\left(d_S(1,w_n) \le \frac{\ell}{2} n \right) \le e^{-n/C}$

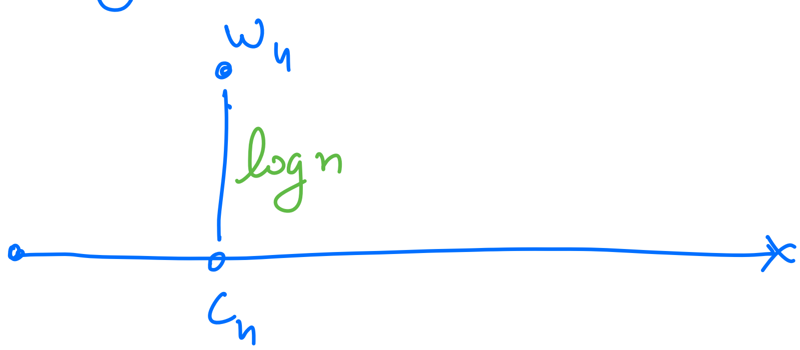
$$n \le \frac{2}{\ell} d_S(1,w_n)$$

$$\boxed{\sup_Y d_Y(1,w_n) \lesssim \log d_S(1,w_n)}$$

<u>hence</u> : $w_n \longrightarrow \xi \in \mathcal{L}_C$

$\Longrightarrow$ g_ξ is K-Morse

<u>Step 4</u>

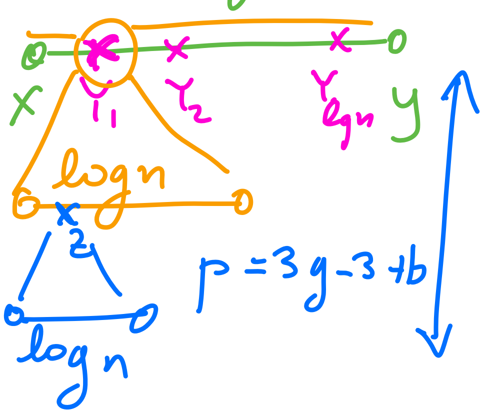


since $d_Y(w_n, c_n) \le \log n \quad \forall Y$

$\Longrightarrow$ distance formula

$$d_G(x,y) = \sum_{Y \subseteq S} \left[d_Y(x,y) \right]_B$$

contributions:



log n $\qquad$ $\mathcal{C}(S)$

log n $\qquad$ at each level of hierarchy

$$\Longrightarrow \quad d_G(w_n, c_n) \lesssim (\log n)^{p+1}$$

$p = 3g-3+b$ $\qquad$ <u>depth of hierarchy</u>

For a.e. (w_n) $\exists \mathcal{G}$ res of a hierarchy in $Mod(S)$ s.t.

$$\overline{\lim_n} \frac{d_G(w_n, \mathcal{G})}{(\log n)^{p+1}} < +\infty .$$

Cor.: Almost every sample path sublinearly tracks a K-Morse quasigeodesic.

Cor.: The RW converges to $\partial_K X$ a.s., and $(\partial_K X, \nu)$ is a model for the Poisson boundary.

Rmk : for rel. hyp. $\to$ same proof
$K(r) = \log(r)$

Rmk : (Sisto) $\to$ $\dfrac{d_G(w_n, \gamma)}{\sqrt{n \log n}} < +\infty$.

THANK YOU !